

广义预变分不等式的 gap 函数

谷爱玲

(广东工业大学 应用数学学院, 广东 广州 510006)

摘要: 定义了广义预变分不等式 (EPVI) 并给出了研究 EPVI 的有力工具——gap 函数.

关键词: 变分不等式; 广义预变分不等式 (EPVI); gap 函数

中图分类号: O224

文献标识码: A

文章编号: 1007-7162(2008)02-0026-03

1 问题的提出

设 K 是欧几里德空间 R^N 中的非空闭凸集, $F: R^N \rightarrow R^N$ 为一连续函数. 所谓的变分不等式问题 (VIP) 是求一点 $x^* \in R^N$, 满足

$$\langle F(x), x - x^* \rangle \geq 0 \quad \forall x \in K.$$

变分不等式问题在控制论、对策论、社会和经济模型等方面都有广泛的应用, 其理论的研究在近几年得到了很好的发展^[1]. 在文献 [2] 中作者将 VIP 推广为预变分不等式 (PVI): K 是 R^N 中的非空闭集, $F: R^N \rightarrow R^N$, $\eta: R^N \times R^N \rightarrow R^N$.

满足 $\eta(x, x) = 0 \quad \forall x \in K$, 预变分不等式 (PVI) 是求一点 $x^* \in K$ 满足

$$\langle F(x^*), \eta(x, x^*) \rangle \geq 0 \quad \forall x \in K.$$

假设 η 是连续函数, 这里给出广义的预变分不等式 (EPVI): 设 $F: R^N \rightarrow R^N$ 是连续可微函数, $\eta: R^N \times R^N \rightarrow R^N$ 是连续可微的且满足 $\eta(x, x) = 0 \quad \forall x \in R^N$; $f: R^N \rightarrow R$ 是实值连续可微函数, 广义的预变分不等式定义如下: 找 $x^* \in R^N$ 使 $\langle F(x^*), \eta(x, x^*) \rangle \geq f(x^*) - f(x), \quad \forall x \in R^N$.

gap 函数是研究 VIP 的一个有力工具, 在文 [3] 中作者为了研究 VIP, 提出了变分不等式的 gap 函数

$$g_{\alpha\beta}(x) = \hat{f}_\beta(x) - \hat{f}_\alpha(x),$$

其中 $\hat{f}_i(x) = \max_{y \in R^N} \langle F(x), x - y \rangle - \phi(x, y), \quad i = \alpha, \beta$
 $\phi: R^N \times R^N \rightarrow R^N$ 满足如下条件:

C_1 : ϕ 在 $R^N \times R^N$ 上连续可微;

C_2 : $\phi(x, y) \geq 0 \quad \forall x, y \in R^N$;

C_3 : $\phi(x, \cdot)$ 在 x 点一致强凸, 即存在 $\lambda > 0$ 使得任意的 $x \in R^N$,

$$\phi(x, y_1) - \phi(x, y_2) \geq$$

$$\langle \tilde{N}_y \phi(x, y_2), y_1 - y_2 \rangle + \lambda \|y_1 - y_2\|^2;$$

$$C_4: \phi(x, y) = 0 \Leftrightarrow x = y;$$

$$C_5: \tilde{N}_x \phi(x, y) = -\tilde{N}_y \phi(x, y), \quad \forall x, y \in R^N;$$

$$C_6: \|\tilde{N}_y \phi(x, y_1) - \tilde{N}_y \phi(x, y_2)\| \leq L_4 \|y_1 - y_2\| \quad \forall y_1, y_2 \in R^N.$$

本文采用类似文 [3] 的方法给出广义预变分不等式 (EPVI) 的 gap 函数.

2 广义预变分不等式的 gap 函数

$$\text{令 } f_\alpha(x) = \min_{y \in R^N} \phi_\alpha(x, y) = \min_{y \in R^N} \langle F(x), \eta(y, x) \rangle -$$

$$f(x) + f(y) + \alpha \phi(x, y),$$

设 $y_\alpha(x)$ 是 $\phi_\alpha(x, y)$ 在 x 点的最小值点. 考虑函数 $g_{\alpha\beta}: R^N \rightarrow R$,

$$g_{\alpha\beta}(x) = f_\beta(x) - f_\alpha(x),$$

其中 $\beta > \alpha > 0$

定理 1 令 ϕ 满足 $C_1 - C_4$, 则 $g_{\alpha\beta}$ 可微, 且

$$\begin{aligned} \tilde{N} g_{\alpha\beta}(x) &= \tilde{N} F(x) (\eta(y_\beta(x), x) - \eta(y_\alpha(x), x)) + \\ &F(x) \left(\frac{\partial \eta(y_\beta(x), x)}{\partial x} - \frac{\partial \eta(y_\alpha(x), x)}{\partial x} \right) + \\ &\beta \tilde{N}_x \phi(x, y_\beta) - \alpha \tilde{N}_x \phi(x, y_\alpha(x)). \end{aligned} \quad (1)$$

证明 由 $F, \eta(\cdot, \cdot)$ 及 $\phi(\cdot, \cdot)$ 的可微性知, $g_{\alpha\beta}$ 可微.

因为

$$\begin{aligned} f_\alpha(z) - f_\alpha(x) &\leq \langle F(z) - F(x), \eta(y_\alpha(x), x) \rangle + \\ &\langle F(z), \eta(y_\alpha(x), z) - \eta(y_\alpha(x), x) \rangle - \\ &f(z) + f(x) + \alpha \phi(z, y_\alpha(x)) - \alpha \phi(x, y_\alpha(x)). \\ f_\alpha(z) - f_\alpha(x) &\geq \langle F(z) - F(x), \eta(y_\alpha(z), x) \rangle + \\ &\langle F(z), \eta(y_\alpha(z), z) - \eta(y_\alpha(z), x) \rangle - \\ &f(z) + f(x) + \alpha \phi(z, y_\alpha(z)) - \alpha \phi(x, y_\alpha(z)), \end{aligned}$$

所以

$$\tilde{N} f_{\alpha}(x) = \tilde{N} F(x) \eta(y_{\alpha}(x), x) + \\ F(x) \frac{\partial \eta(y_{\alpha}(x), x)}{\partial x} - \tilde{N} f(x) + \alpha \tilde{N}_x \phi(x, y_{\alpha}(x)),$$

故

$$\tilde{N} g_{\alpha\beta}(x) = \tilde{N} F(x) (\eta(y_{\beta}(x), x) - \eta(y_{\alpha}(x), x)) + \\ F(x) \left(\frac{\partial \eta(y_{\beta}(x), x)}{\partial x} - \frac{\partial \eta(y_{\alpha}(x), x)}{\partial x} \right) + \\ \beta \tilde{N}_x \phi(x, y_{\beta}(x)) - \alpha \tilde{N}_x \phi(x, y_{\alpha}(x)). \\ C'_5: \tilde{N}_x \phi(x, y) = -\tilde{N}_y \phi(x, y); \\ \frac{\partial \eta(y, x)}{\partial x} = -\frac{\partial \eta(y, x)}{\partial y}.$$

推论 1 如果 ϕ 满足 $C_1 - C_4, C'_5$, 则

$$\tilde{N} g_{\alpha\beta}(x) = \tilde{N} F(x) (\eta(y_{\beta}(x), x) - \eta(y_{\alpha}(x), x)) + \\ \tilde{N} f(y_{\beta}(x)) - \tilde{N} f(y_{\alpha}(x)).$$

命题 1 函数 $g_{\alpha\beta}$ 具有如下性质

$$(\beta - \alpha) \phi(x, y_{\beta}(x)) \leq g_{\alpha\beta}(x) \leq \\ (\beta - \alpha) \phi(x, y_{\alpha}(x)).$$

证明 由

$$g_{\alpha\beta}(x) \leq \langle F(x), \eta(y_{\alpha}(x), x) - \eta(y_{\alpha}(x), x) \rangle + \\ f(y_{\alpha}(x)) - f(y_{\alpha}(x)) + (\beta - \alpha) \phi(x, y_{\alpha}(x)) = \\ (\beta - \alpha) \phi(x, y_{\alpha}(x)),$$

同理可得

$$y_{\alpha\beta}(x) \geq (\beta - \alpha) \phi(x, y_{\beta}(x)).$$

命题 2 如果 ϕ 满足 $C_1 - C_4, C'_5$, 则 $\tilde{N}_y \phi(x, y) = 0$ 的充要条件是 $x = y$

证明略 (类似文献 [3] 的引理 3.1)

定义 1 令 $f: R^N \rightarrow R, \eta: R^N \times R^N \rightarrow R^N$ 满足 $\eta(x, x) = 0$ 这里称 f 关于 η 是预凸的, 只要满足

1) 任意 $x, y \in \text{dom}(f)$, 有 $x + t\eta(y, x) \in \text{dom}(f)$;

2) $f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y), \forall x, y \in \text{dom}(f)$.

如果 f 是预凸, 则当 f 可微时有

$$f(y) \geq f(x) + \langle \tilde{N} f(x), \eta(y, x) \rangle \\ \forall x, y \in \text{dom}(f).$$

假设 $A_1: \frac{\partial \eta(x, x)}{\partial y} = I$ 且 f 是关于 η 预凸的.

定义 2 函数 $\phi: K \rightarrow R$ 被称为 $VI(K, F)$ 的 gap 函数, 如果

(i) $\phi(x) \leq 0$

(ii) $\phi(x_0) = 0 \Leftrightarrow x_0$ 是 $VI(K, F)$ 的解.

定理 2 $-g_{\alpha\beta}$ 是 EPV I 的 gap 函数.

证明 由 C_2 知 $\phi(x, y) \geq 0, \forall x, y \in R^N$, 从而 $g_{\alpha\beta} \geq 0$ 下面分两步证明:

(i) $x = y_{\alpha}(x) = y_{\beta}(x)$ 的充要条件是 $g_{\alpha\beta}(x) = 0$

必要性显然成立. 只需证充分性. 由 $g_{\alpha\beta}(x) = 0$ 则 $\phi(x, y_{\beta}(x)) = 0$ 从而 $x = y_{\beta}(x)$, 进一步 $f_{\beta}(x) = f_{\alpha}(x) = 0$ 由假设 y_{α} 是 $\phi_{\alpha}(x, y)$ 的唯一最小值点, 故 $x = y_{\alpha}(x) = y_{\beta}(x)$.

(ii) $x = y_{\alpha}(x) = y_{\beta}(x)$ 的充要条件是 x 是 EPV I 的解.

如果 x 为 EPV I 的解, 则 $\langle F(x), \eta(y, x) \rangle - f(x) + f(y) \geq 0, \forall y \in R^N$.

而

$$f_{\alpha}(x) = \langle F(x), \eta(y_{\alpha}(x), x) \rangle - \\ f(x) + f(y_{\alpha}(x)) + \alpha \phi(x, y_{\alpha}(x)),$$

由 C_2 知, $f_{\alpha}(x) \geq 0$

但另一方面,

$$f_{\alpha}(x) = \min_{y \in R^N} \langle F(x), \eta(y, x) \rangle - f(x) + \\ f(y + \alpha \phi(x, y)) \leq 0$$

故 $f_{\alpha}(x) = 0$ 同理, $f_{\beta}(x) = 0$ 进一步得到 $x = y_{\alpha}(x) = y_{\beta}(x)$.

反过来, 由 $x = y_{\alpha}(x)$ 且 $y_{\alpha}(x)$ 是 $\phi_{\alpha}(x, y)$ 的最小值点, 得

$$\tilde{N}_y \phi_{\alpha}(x, y_{\alpha}(x)) = 0$$

即

$$0 = F(x) + \tilde{N} f(x),$$

$$0 = \langle F(x) + \tilde{N} f(x), \eta(y, x) \rangle \leq \langle F(x), \eta(y, x) \rangle + \\ f(y) - f(x), \forall y \in R^N.$$

因此由 (i) (ii) 知: $g_{\alpha\beta}(x) = 0$ 的充要条件是 EPV I 有解, 所以 $-g_{\alpha\beta}$ 是 EPV I 的 gap 函数.

定义 3 1) 设 $F: R^N \rightarrow R^N, \eta: R^N \times R^N \rightarrow R^N$, 称 $\tilde{N} F$ 关于 η 是正定的, 如果任意的 $y, z \in R^N$, 满足

$$\langle y - z, \tilde{N} F(x) (\eta(y, x) - \eta(z, x)) \rangle > 0, y \neq z$$

2) 设 $F: R^N \rightarrow R^N, \eta: R^N \times R^N \rightarrow R^N$, 称 F 关于 η 是强单调, 如果 $\forall y, z \in R^N$ 且 $y \neq z$ 则有

$$\langle F(y) - F(z), \eta(y, x) - \eta(z, x) \rangle \geq \mu \|y - z\|^2.$$

注: 当 $\eta(y, x) = y - x$ 时, $\tilde{N} F$ 是一般意义上的正定, F 也是一般意义上的强单调.

定理 3 如果 ϕ 满足 $C_1 - C_4, C'_5$ 且函数 $\tilde{N} f$ 是单调函数, 而且 $\tilde{N} F$ 关于 η 是正定的, 则当 x 为 $g_{\alpha\beta}$ 的稳定点时, 有 $x = y_{\alpha}(x) = y_{\beta}(x)$.

证明 由推论 1 知, 当 $\tilde{N} g_{\alpha\beta}(x) = 0$ 时, 有

$$\tilde{N} F(x) (\eta(y_{\beta}(x), x) - \eta(y_{\alpha}(x), x)) + \\ \tilde{N} f(y_{\beta}(x)) - \tilde{N} f(y_{\alpha}(x)) = 0$$

于是

$$\langle \tilde{N} F(x) (\eta(y_{\beta}(x), x) - \eta(y_{\alpha}(x), x)) + \\ \tilde{N} f(y_{\beta}(x)) - \tilde{N} f(y_{\alpha}(x)), y_{\beta}(x) - y_{\alpha}(x) \rangle = 0$$

由 $\tilde{N} F$ 关于 η 是正定及 $\tilde{N} f$ 单调知

$$\langle y_{\beta}(x) - y_{\alpha}(x), \tilde{N}F(x) (\eta(y_{\beta}(x), x) - \eta(y_{\alpha}(x), x)) \rangle \geq 0$$

$$\langle y_{\beta}(x) - y_{\alpha}(x), \tilde{N}f(y_{\beta}(x)) - \tilde{N}f(y_{\alpha}(x)) \rangle \geq 0$$

于是

$$y_{\beta}(x) = y_{\alpha}(x). \quad (2)$$

将式 (2) 代入定理 1 的 $\tilde{N}g_{\alpha\beta}(x) = 0$ 得到 $\tilde{N}_y\phi(x, y_{\alpha}(x)) = 0$ 由命题 2 得 $x = y_{\alpha}(x) = y_{\beta}(x)$.

参考文献:

[1] Harker P T, Pang J S. Finite dimension Variational inequality

ty and non linear complementarity problems: A survey of theory, algorithm, and applications[J]. Mathematical Programming 1990, 48(2): 161-220

[2] Yang X Q, Chen G Y. A class of nonconvex functions and variational inequalities[J]. Journal of Mathematical Analysis and Applications 1992, 169(2): 359-373

[3] Nobuo Yamashita Kou ichi Tajiri Masao Fukushima. Uncircstrained optimization reformulations of variational inequality problems[J]. J. Optim. Theory Appl 1997, 92(3): 439-456

A Gap Function for Solving the Problem of Extended Prevariational Inequality

Gu A r ling

(Faculty of Applied Mathematics, Guangdong University of Technology, Guangzhou 510006, China)

Abstract Extended Prevariational Inequality is defined, and a gap function is provided for solving it

Key words variational inequality; extended prevariational inequality; gap function

(上接第 21 页)

A Study of the Effects of Cigarette Materials on the Degree of Ventilation and Filtration Efficiency of the Cigarette with 24 mm Filter Rod

Wei Yu-ling, Wang Limin, Hu Qun, Feng Wen, Liao Zhen, Sun Zhiyong

(Yunnan Academy of Tobacco Science, Kunming 650106, China)

Abstract The orthogonal test, which dealt with different cigarette Paper, tipping paper, wrapping paper and filter, was designed and applied in situ. Test was to study the effects of the above factors on the degree of ventilation and filtration efficiency of the cigarette with 24mm filter rod. The result indicates that the air permeability of tipping paper, wrapping paper, cigarette paper and the pressure drop of filter rod are the most prominent and important factors affecting the degree of ventilation and filtration efficiency. Therefore, the adjustment and control of these factors are the keystones of the cigarette quality control, which is one of the best ways to reduce the costs of the products. Compared with the other methods, this design of the experiment is shortcut, efficient, economical and veracious. By this method, the design and the techniques of the aimed product are optimized.

Key words degree of ventilation; filtration efficiency; Cigarette materials; orthogonal test